RELATIVITY AND COSMOLOGY I

Problem Set 3 Fall 2023

1. Spherical Polar Coordinates

Consider $\mathbb{R}^3$ as a manifold with the flat Euclidean metric, and coordinates $\{x, y, z\}$. We will introduce spherical coordinates $\{r, \theta, \phi\}$, which are related to $\{x, y, z\}$ through

$$x = r\sin\theta\cos\phi\,, \qquad y = r\sin\theta\sin\phi\,, \qquad z = r\cos\theta\,. \tag{1}$$

with

$$r \in [0, \infty), \qquad \theta \in [0, \pi]\,, \qquad \phi \in [0, 2\pi)\,. \tag{2}$$

(a) Show that the flat Euclidean metric in spherical coordinates takes the form

$$ds^2 = \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2\,. \tag{3}$$

(b) A particle moves along a parametrized curve given by

$$x(\lambda) = \cos\lambda\,, \qquad y(\lambda) = \sin\lambda\,, \qquad z(\lambda) = \lambda\,. \tag{4}$$

Express the path of the curve in the $\{r, \theta, \phi\}$ system.

(c) Calculate the components of the tangent vector to the curve in both the Cartesian and spherical polar coordinate systems.

2. Rindler Coordinates

(a) Prove that a particle starting at rest in a uniform electric field $\vec{E}$ along the x axis has the following trajectory

$$x(t) = x_0 + \frac{1}{a}\left(\sqrt{1 + a^2t^2} - 1\right)\,. \tag{5}$$

Hint: Start with the equations for the dynamics of a particle in an electromagnetic field

$$\frac{\mathrm{d}p^\alpha}{\mathrm{d}\tau} = qF^{\alpha\beta}U_\beta\,. \tag{6}$$

and use the mass shell condition.

(b) Show that, for a suitable choice of the integration constant x_0, the above trajectory can be expressed as

$$x^2 - t^2 = \frac{1}{a^2}\,, \quad \longrightarrow \quad x = \frac{1}{a}\cosh(\alpha)\,, \qquad t = \frac{1}{a}\sinh(\alpha)\,. \tag{7}$$

(c) Show that the proper time τ of a rocket moving on such a trajectory is given by

$$\tau = \frac{1}{a}\mathrm{arcsinh}(at)\,, \tag{8}$$

and relate the parameter α to τ. What happens in the $a \to 0$ limit?

Now we want to find the relation between the coordinates (t, x) that an observer in the laboratory frame uses to describe some event P and the coordinates (t', x') that an observer on an accelerated rocket uses to describe the same event P. The rocket observer measures (t', x') by sending (event A) a light signal to P, where it reflects and then eventually returns (event B) to the rocket observer. The rocket observer defines her coordinates through

$$t' = \frac{t'_A + t'_B}{2}\,, \qquad x' = \frac{t'_B - t'_A}{2}\,. \tag{9}$$

(d) Argue in favor of her definitions.

On the other hand, the laboratory observer measures

$$x - x_A = t - t_A\,, \qquad x - x_B = t_B - t\,. \tag{10}$$

(e) Show that, for a rocket going on a generic motion $t = f_0(\tau)$ and $x = f_1(\tau)$, the measurements of the two observers are related through

$$\begin{aligned} t - x &= f_0(t' - x') - f_1(t' - x')\,, \\ t + x &= f_0(t' + x') + f_1(t' + x')\,. \end{aligned} \tag{11}$$

(f) Show that, for the specific case of a uniformly accelerated rocket, this implies that

$$\begin{aligned} x &= \frac{1}{a}\cosh(at')e^{ax'}\,, \\ t &= \frac{1}{a}\sinh(at')e^{ax'}\,. \end{aligned} \tag{12}$$

(g) Verify that these transformation laws are consistent with the motion of a uniformly accelerated particle as studied in the first part of this exercise.[1]

(h) Write the line element in the coordinates (t', x'). This is the Rindler line element.

(i) Find the patch covered by (t', x') coordinates in terms of (t, x).

(j) Working in (t, x) coordinates, show that light signals emitted at $t = 0$ from some position $x < 0$ will never reach the rocket (use the choice of x_0 from your answer to question (b)). Can you see this fact by drawing a spacetime diagram?

3. Surface - 2022 Exam exercise

Consider a two dimensional surface with the following metric

$$ds^2 = \left(1 + \frac{r^2}{a^2}\right)\mathrm{d}r^2 + r^2\mathrm{d}\phi^2\,, \tag{13}$$

with $r > 0$ and $\phi \in [0, 2\pi]$.

[1] For example check that $t'|_{x'=0} = \tau$.

(a) What is the length of the curve C defined by the equation $r = R$?

(b) What is the area inside the curve C?

(c) What is the distance from the point $r = 0$ to the curve C?
Hint: the following integral may be useful $\int dx \cosh^2 x = \frac{x}{2} + \frac{1}{4}\sinh(2x)$.

(d) Show that this surface can be isometrically embedded as a paraboloid in $\mathbb{R}^3$.